



## Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).

# ***Notes.***

## I.

### *On Symbols of Operation.*

BY PROFESSOR CROFTON, F. R. S.

To prove that,  $\phi$  being any function of  $D$ , i. e.  $\frac{d}{dx}$ ,

$$e^{x\phi(D)} e^{hx} = e^{\lambda x}, \quad (1)$$

where  $\lambda$  is a constant determined from

$$\psi(\lambda) = 1 + \psi(h), \quad (2)$$

the function  $\psi$  being defined by

$$\psi(x) = \int \frac{dx}{\phi(x)}. \quad (3)$$

The above theorem might be made to follow from principles given in a paper by me in the Proceedings of the London Mathematical Society, April 1881; but the following method may be also employed.

Let  $u = e^{kx\phi(D)} e^{hx}$ ;

differentiating with regard to  $h$ ,

$$\frac{du}{dh} = e^{kx\phi(D)} x e^{hx},$$

also

$$\begin{aligned} \frac{du}{dk} &= e^{kx\phi(D)} x \phi(D) e^{hx} \\ &= \phi(h) e^{kx\phi(D)} x e^{hx}. \end{aligned}$$

We have thus a partial differential equation

$$\frac{du}{dk} = \phi(h) \frac{du}{dh}$$

$$\therefore u = \chi \left( k + \int \frac{dh}{\phi(h)} \right) = \chi \left( k + \psi(h) \right).$$

To determine the arbitrary function  $\chi$ , we observe that if  $k = 0$ ,  $u = e^{hx}$ ;

$$\therefore \chi(\psi h) \equiv e^{hx}$$

hence if  $\lambda$  be determined so as to satisfy

$$\begin{aligned} \psi(\lambda) &\equiv k + \psi(h) \\ u &= \chi(\psi\lambda) = e^{\lambda x}. \end{aligned}$$

For instance, let  $\phi(D) = D^2$ ;  $\psi(x) = -\frac{1}{x}$  by (3); hence by (2),  $\lambda = \frac{h}{1-h}$ ;

$$\therefore e^{xD^2} e^{hx} = e^{\frac{hx}{1-h}}$$

Again

$$e^{kxD^{-1}} e^{hx} = e^{x\sqrt{h^2+2k}}.$$

Again

$$e^{kxD^r} e^{hx} = e^{\lambda x},$$

where

$$\lambda = [h^{1-r} + (1-r)k]^{\frac{1}{1-r}}.$$

The above process may sometimes be applied in other similar cases; for example, to find

$$u = e^{kx^{-1}D^2} e^{hx^3};$$

we may deduce the equation

$$\frac{du}{dk} = 9h^2 \frac{du}{dh} + 6hu;$$

the solution of this by Lagrange's method, or otherwise, is

$$u = h^{-\frac{2}{3}} \phi(h^{-1} - 9hk);$$

and determining  $\phi$  from the condition that  $u = e^{hx}$  when  $k = 0$ , we find

$$u = (1 - 9hk)^{-\frac{2}{3}} \exp. \left( \frac{hx^3}{1 - 9hk} \right).$$

A very remarkable identity between certain symbolic operators of the exponential form may be established by a comparison of Lagrange's theorem and a result which I have given in the paper above referred to. It is there shown that if  $z$  is related to  $x$  by the equation

$$\psi(z) = 1 + \psi(x) \dots (1)$$

and if we put for shortness

$$\frac{1}{\psi'(x)} = \phi(x) = \phi \dots (2)$$

then

$$e^{\phi D} F(x) = F(z) \dots (3)$$

also

$$e^{D\phi} F(x) = F(z) \frac{dz}{dx} \dots (4)$$

whatever be the function  $F$ .

Now if we use for shortness the symbol  $e^{D \cdot f(x)}$  or  $e^{D \cdot f}$  to express the development

$$e^{D \cdot f} \equiv 1 + Dfx + \frac{1}{1 \cdot 2} D^2(fx)^2 + \&c., \dots (5)$$

Lagrange's theorem gives

$$e^{D \cdot f} F(x) \equiv F(z) \frac{dz}{dx} \dots (6)$$

where

$$z = x + f(z) \dots (7)$$

Hence if  $z$  be the same function of  $x$  in (6) as in (4), the operator  $e^{D\phi}$  is identical with  $e^{D\cdot f}$ ; that is

$$1 + D\phi + \frac{1}{1.2} \overline{D\phi}^2 + \&c. \equiv 1 + Df + \frac{1}{1.2} D^2 f^2 + \&c. \dots (8)$$

In order that this identity should hold, it is necessary that  $z$ ,  $f(x)$ ,  $\psi(x)$  shall be three functions of  $x$  which satisfy (1) and (7), viz.

$$z = x + f(z) \dots (7)$$

$$\psi(z) = 1 + \psi(x) \dots (1)$$

$\phi(x)$  being put for  $\frac{1}{\phi'(x)}$ .

If we could then eliminate  $(z)$  from (1) and (7), any functions  $f$ ,  $\psi$ , whose forms satisfy the resulting equation, will cause the identity (8) to hold.

For instance, suppose  $\phi(x) = hx^2$ , then  $\psi x = -\frac{1}{hx}$ ;

hence (1)

$$x = \frac{z}{1 + hz}$$

therefore (7)

$$f(z) \equiv \frac{hz^2}{1 + hz}$$

therefore  $1 + hDx^2 + \frac{h^2}{1.2} \overline{Dx^2}^2 + \&c. \equiv 1 + hD \frac{x^2}{1 + hx} + \frac{h^2}{1.2} D^2 \left( \frac{x^2}{1 + hx} \right)^2 + \&c.$   
whatever be the operand.

Again let us suppose  $f(x) = hx$ , then  $z = \frac{x}{1 - h}$ ;

hence (7)  $\psi$  is to be found from

$$\psi\left(\frac{x}{1-h}\right) \equiv 1 + \psi(x)$$

$$\therefore \psi(x) \equiv -\frac{\log x}{\log(1-h)}$$

$$\therefore \phi(x) = -x \log(1-h)$$

Thus

$$1 + hDx + \frac{h^2}{1.2} D^2 x^2 + \dots \equiv e^{-\log(1-h)Dx} \equiv (1-h)^{-Dx}$$

$$\equiv 1 + hDx + \frac{h^2}{1.2} Dx(Dx+1) + \frac{h^3}{1.2.3} Dx(Dx+1)(Dx+2) + \dots$$

This is easy to verify.